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**Article**

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# Revealed Preferences

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**Abstract:** *When the preferences of a consumer can be represented by a utility function, the consumption (the demand) of the consumer is obtained by the maximization of the utility under the consumer budget constraint and the prices of goods. Actually, what is known is the demand correspondence obtained from observations and not one hypothetical utility function. The revealed preferences consists in constructing utility functions from demand correspondences. We present a short state of art on the question in the differentiable case as well in the non differentiable case. Things are not so simple as shown from some pathological examples.*

**Keywords:** *Consumer Theory; Cyclic Monotony; Preferences Order; Utility Function*.

**Classificação JEL:** C02; D11.

## 1. Introduction

Consumer theory is concerned with the analysis of the behavior of a consumer when facing his budget and the prices of goods (or services). It is assumed that the consumer is able to make comparisons between two different choices in the set of available goods and he is free of making his choice in function only of his preferences and his budget constraint. Then, under rational preferences, the set of goods can be ordered, an utility function, which measures the satisfaction of the consumer when taking his choice, is associated to the order. Then, the consumer problem reduces to the maximization of the utility function under the budget constraint.

The real problem consists in building the preferences order. The only known material is the result of observations on the consumptions, not a hypothetical order, not a utility function.

The revealed preferences consists in the construction of the preferences order from the observations on the consumptions of the consumer, in fact in constructing a utility function representing the preorder.

## 2. Preferences order and utility functions: the primal side

In this simplified presentation, the set *K* of goods is the nonnegative orthant of the euclidean space $IR_+^d$, *K* is ordered with symbols $\prec, \preceq$ and $\simeq$. For all $x, y \in K$,

1. One and only one of the three following assertions holds: $x \prec y$ (*y* is strictly preferred to *x*), $y \prec x$ (*x* is strictly preferred to *y*), $x \simeq y$ (*x*,*y* are equally preferred).

2. If $x_i \leq y_i$ for all *i* and $x \neq y$, then $x \prec y$.

3. Let $0 < t < 1$, then :

a) If $x \preceq y$, then $x \preceq x + t(y - x)$

b) If $x \prec y$, then $x \prec x + t(y - x)$.

4. If $x \preceq y \preceq z$, then $x \preceq z$.

5. If $x \preceq y \prec z$ or $x \prec y \preceq z$, then $x \prec z$.

$x \in K$ being given, the sets

$$S_x = \{y \in K : x \preceq y\} \ and \ S_x^s = \{y \in K : x \prec y\}$$

are convex in reason of 3a and 3b.

Assume in addition that the sets $S_x$ are closed and the sets $S^s{}_x$ are open. A fundamental of Debreu [1] says that there exists a continuous function *u*, called *a utility function* such that

$$u(x) \leq u(y) \Leftrightarrow x \preceq y \ and \ u(x) < u(y) \Leftrightarrow x \prec y.$$

By construction *u* is strictly increasing and quasiconcave. The function *u* is said to be a representation of the preorder. This representation is not unique: if *u* represents the preorder, any function of type $k \circ u$, with *k* continuous strictly increasing, is an equivalent representation.

## 3. On the dual side

Given the consumption $x \in K$ and the vector of prices $\pi \in IR_+^d$, the cost of $x$ is $\pi^t x = \pi_1 x_1 + \pi_2 x_2 + \cdots + \pi_d x_d$ where $x_i$ and $\pi_i$ are respectively the quantity and the unitary price of the good of type *i*. Assume that the budget of the consumer is $w > 0$, the consumer problem consists in choosing some $x$ within the set of the best possible choices $X(\pi, w)$

$$X(\pi, w) = \{x \in K : x \prec y \Rightarrow \pi^t y > \pi^t x = w\}.$$

It is practical to normalize the prices (think of different currencies for example). Set $p = \pi / w, X(\pi, w) = X(p,1) = X(p).$

When $u$ is a utility function associated to the preorder, the ordinal formulation of the consumer problem is equivalent to the cardinal formulation: find $x \in X(p)$ where

$$X(p) = \arg\max_x [u(x) : p^t x \leq 1].$$

The correspondance $X$ is called *demand*.

The cardinal formulation leads to the introduction of the function

$$v(p) = \max_x [u(x) : p^t x \leq 1].$$

Under mild assumptions, there is a quite symmetric duality (Lau [2], Diewert [3], Crouzeix [4] , Martinez-Legaz [5],....) between the direct utility function $u$ and its indirect utility function $v$. The function $v$ is strictly decreasing and quasiconvex,

$$x \in X(p) \Leftrightarrow [u(x) = v(p)\, and\ p^t x = 1] \Leftrightarrow p \in P(x),$$
$$with\ P(x) = \arg\min_p [v(p) : p^t x \leq 1].$$

The revealed preferences problem (Samuelson [6], Houthakker [7], .....) consists in building the preferences order via its representation by a direct utility function $u$ (or, in reason of the duality, via the indirect utility function $v$ associated to $u$) from the observations on the demand $X$.

## 4. The case where $X$ is univalued and continuously differentiable

The dual side of the problem is the more appropriate: does there exist a quasiconvex, differentiable function $v$ so that *X(p)* is colinear to $\nabla v(p)$ ? It is easily seen that the following condition is necessary:

$$The\ matrix\ X^{'}(p)\ is\ positive\ semi-definite\ on\ [X(p)]^{\perp}. \qquad \text{(CN)}$$

What about sufficiency?

*Case* n = 2: It was known in the early fifties that (CN) is also a sufficient condition (Samuelson). Remark that in this case the dimension of the space $[X(p)]^{\perp}$ is 1.

*Case* n > 2: The necessary and sufficient condition is

$$The\ matrix\ X^{'}(p)\ is\ psd\ and\ symmetric\ on\ [X(p)]^{\perp}. \qquad \text{(CNS)}$$

The dimension of the space $[X(p)]^{\perp}$ is greater or equal to 2, the problem becomes very hard. There are two types of proofs:

1) By construction of the indifference curves:

i) Crouzeix-Rapcsak [8], 2005, with a very "handmade" proof.

ii) Penot-Hadjisavvas [9], 2015, with a more scholar proof based on the Frobenius theorem.

2) Symplectic geometry:

Chiappori-Ekeland [10], 1999, using differential exterior calculus, Darboux theorem.

## 5. Cyclic pseudomonotony

Let us place in the situation where $X$ is the demand correspondence associated with the utility function *u*.

For all finite sequence $\{(x_i, p_i)\}_{i=0}^{q} \subset gph(X)$ so that $(x_0, p_0) = (x_q, p_q)$ and $p_i^t(x_{i+1} - x_i) \leq 0$ for all *i* one has:

*either* $u(x_0) = u(x_q)$. Then, $u(x_i) = u(x_0)$ and $p_i^t(x_{i+1} - x_i) = 0$ for all *i*.

*either* some *j* exists so that $u(x_{j+1}) > u(x_j)$. Then $p_j^t(x_{j+1} - x_j) > 0$ and finally $\max_i p_i^t(x_{i+1} - x_i) > 0$.

For mathematicians, X is said to be cyclic pseudomonotone, the economists (Houthakker [7], Samuelson [6], Varian [11], .... ) speak of axioms of revealed preferences (GARP, SARP, CARP).

## 6. The Afriat's construction

Let us place in the case where the point to set map *X* is cyclic pseudomonotone and the family $\{(x_j, p_j)\}_{j \in J} \subset gph(X)$ is finite.

Then, Afriat [12] (see also Diewert [13], Fostel-Scarf-Todd [14]), shows the existence of positive numbers $\alpha_j$, $\beta_j > 0$ such that the piecewise linear function $u_J$ defined by

$$u_J(x) = \min_j [\alpha_j + \beta_j p_j^t(x - x_j)] \, \forall x$$

is concave and such that for all $j \in J, u_J(x_j) = \alpha_j$ and $\beta_j p_j$ belongs to the upper-gradient $\partial^s(u_J)(x_j)$ of the concave function $u_J$ at $x_j$.

The numbers $\alpha_j$ and $\beta_j$ are the optimal solutions of a linear programming problem, the feasibility of this program being equivalent to the cyclic pseudomonotonicity of *X*.

The order induced by $u_J$ depends on the objective function of the linear program. It results that the same family *J* can birth two different approximated orders. Furthermore, different families lead to different approximated orders.

How to proceed to comparisons? The response is by rescalarizations of the approximated utility functions. Proceed as follows:

Set $e = (1, 1, \cdots, 1) \in IR^d$ and $k_J(t) = u_J(te)$ for all *t*. By construction $k_J$ is concave, continuous and strictly concave.

Next, take $\tilde{u}_J = [k_J]^{-1} \circ u_J$. By construction $\tilde{u}_J$ is pseudoconcave and $\tilde{u}_J(te) = t$ for all *t*. Moreover,

$$\tilde{u}_J(y) \geq \tilde{u}_J(x_j) \Leftrightarrow p_j^t(y - x_j) \geq 0 \quad \forall_j, \forall_y.$$

What happens for $\tilde{u}_J$ when the size of *J* grows? for the limit $\tilde{u}$ (if such a limit exists)?

when the limit is not concavifiable (of the form $k^{-1} \circ g$ with *g* concave)?

## 7. Sandwich inequalities, the finite case

Assume that *X* and *J* are given as in the Afriat's construction.

Denote by $\upsilon_J$ be the class of nondecreasing quasiconcave functions *u* on *K* such that

$$u(te) = t \ \forall t > 0 \ \textit{and} \ x_j \in \arg\max_{y \in K}[u(y) : p_j^t(y - x_j) \le 0] \ \forall j \in J.$$

Then, (Crouzeix-Keraghel-Rahmani [15]),

$$\exists u_J^-, u_J^+ \in \upsilon_J \ \textit{such that} \ u_J^- \le u \le u_J^+ \ \ \forall u \in \upsilon_J.$$

The functions $u^-_J$ and $u^+_J$ are built via rather easy Operations Research technics. The construction provides a test for the cyclic monotony of *X*. The method is competitive with the Afriat's construction. More important,

$$J_1 \subset J_2 \Rightarrow u_{J_1}^- \le u_{J_2}^- \le u_{J_2}^+ \le u_{J_1}^+.$$

## 8. Sandwich inequalities, the general case

Let *X* be cyclic pseudomonotone on *K*.

Let $\upsilon$ be the class of nondecreasing quasiconcave functions *u* on *K* such that

$$u(te) = t \ \forall t > 0 \ \textit{and} \ X(p) \subset \arg\max_{y \in K}[u(y) : p^t y \le 1] \ \forall p.$$

Set for all $x \in K$

$$J(x) = \left\{ y \in K : \begin{array}{c} \exists k \in \mathbb{N}, \ x_0 = x, \ x_k = y, \\ (x_0, p_0), \cdots, (x_k, p_k) \in gph(X), \\ p_i^t(x_{i+1} - x_i) \le 0, \forall_i = 0, \cdots, p-1 \end{array} \right\}.$$

Next, let us define the functions $u^-$ and $u^+$ on *K* by

$$u^-(x) = \sup_t [t : te \in J(x)] \ \textit{and} \ u^+(x) = \inf_t [t : x \in J(te)].$$

Then, $u^-$ and $u^+$ are quasiconcave and nondecreasing (Crouzeix-Eberhard-Ralph [16]). Both belong to $\upsilon$ and

$$u^- \le u \le u^+ \ \ \forall u \in \upsilon.$$

What conditions on *X* to be added in order that $u^- = u^+$ and $X(p) = \arg\max_{y \in K}[u(y) : p^t y \le 1]$ for all *p*?

There is a counter-example where *X* is cyclic pseudomonotone, maximal pseudomonotone, $u^-$ and $u^+$ coincide but are not pseudoconcave.

In another counter-example, *X* is cyclic pseudomonotone, maximal pseudomonotone but $u^- \neq u^+$. This means that there are different orders sharing the same demand.

Maximal cyclic pseudomonotonicity is not enough. Report to Crouzeix-Eberhard-

Ralph [16].

## References

[1] Debreu G.: Theory of Value: An Axiomatic Analysis of Economic Equilibrium, New York: John Wiley and Sons, Inc.; London: Chapman and Hall, Ltd., 1959

[2] Lau L. J.: Duality and the structure of utility functions, Journal of Economic Theory 1, 374–396,1970.

[3] Diewert W. E.: Duality approaches to microeconomics theory, in Handbook of Mathematical Economics vol. 2, Arrow K. J. and Intrilligator M. D. eds, North Holland Publishing Company, Amsterdam, 535–599, 1982.

[4] Crouzeix J.-P.: Duality between direct and indirect utility functions, differentiability properties, J. Math. Econom. 12, 149–165 ,1983.

[5] Martinez-Legaz J.-E.: Duality between direct and indirect utility functions under minimal hypotheses, J. Math. Econom. 20, 199–209, 1991.

[6] Samuelson, P. A.: Consumption theory in terms of revealed preference, Economica, 243–253, 1948.

[7] Houthakker, H. S.: Revealed preference and the utility function, Economica 17, 1950, 159–174.

[8] Crouzeix J.-P., Rapcsák T.: Integrability of pseudomonotone differentiable maps and the revealed preference problem, J. Convex Anal. 12, no 2, 431–446 (2005).

[9] Hadjisavvas N., Penot J.-P.: Revisiting the problem of integrability in utility theory, Optimization 64, 2495–2509, 2015.

[10] Chiappori P.A., Ekeland: I., Aggregation and market demand: An exterior differential calculus viewpoint, Econometrica 67, 1435–1458, 1999.

[11] Varian, H. R.: The non-parametric approach to demand analysis. Econometrica 50, 945–974, 1982.

[12] Afriat, S. N.: The construction of a utility function from expenditure data, International Economic Review 8, 67–77, 1967.

[13] Diewert, E.: Afriat and revealed preference theory, Review of Economic Studies 40, 419–426, 1973.

[14] Fostel A., Scarf H. E., Todd M. J.: Two new proofs of Afriats theorem, Economic Theory 24, 211–219, 2004.

[15] Crouzeix J.-P., Keraghel A., Rahmani, N. Integration of pseudomonotone maps and the revealed preference problem , Optimization 60, 783– 800, 2011.

[16] Crouzeix J.-P., Eberhard A., Ralph D. (Convex) Level Sets Integration, Journal of Optimization Theory and Applications 171 (3), 865–886, 2016.

[17] Rockafellar R. T., Variational Analysis of Preference Relations, University of Washington, 2022.