

DIGITALES ARCHIV

ZBW – Leibniz-Informationszentrum Wirtschaft
ZBW – Leibniz Information Centre for Economics

Ioan, Cătălin Angelo; Ioan, Gina

Article

An unified consumption and production model for a closed economy

Provided in Cooperation with:

Danubius University of Galati

Reference: Ioan, Cătălin Angelo/Ioan, Gina (2018). An unified consumption and production model for a closed economy. In: The journal of accounting and management 8 (2), S. 5 - 10.

This Version is available at:

<http://hdl.handle.net/11159/3070>

Kontakt/Contact

ZBW – Leibniz-Informationszentrum Wirtschaft/Leibniz Information Centre for Economics
Düsternbrooker Weg 120
24105 Kiel (Germany)
E-Mail: [rights\[at\]zbw.eu](mailto:rights[at]zbw.eu)
<https://www.zbw.eu/econis-archiv/>

Standard-Nutzungsbedingungen:

Dieses Dokument darf zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden. Sie dürfen dieses Dokument nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, aufführen, vertreiben oder anderweitig nutzen. Sofern für das Dokument eine Open-Content-Lizenz verwendet wurde, so gelten abweichend von diesen Nutzungsbedingungen die in der Lizenz gewährten Nutzungsrechte.

<https://zbw.eu/econis-archiv/termsfuse>

Terms of use:

This document may be saved and copied for your personal and scholarly purposes. You are not to copy it for public or commercial purposes, to exhibit the document in public, to perform, distribute or otherwise use the document in public. If the document is made available under a Creative Commons Licence you may exercise further usage rights as specified in the licence.

An Unified Consumption and Production Model for a Closed Economy

Cătălin Angelo Ioan¹, Gina Ioan²

Abstract: The article presents an unified consumption and production model for a closed economy.

Keywords: consumption, production, utility

JEL Code: E17; E27

1. Introduction

Let consider n goods: G_1, \dots, G_n whose elasticity of utility is constant, their prices being p_1, \dots, p_n . For a consumer whose available income is V , the utility function corresponding to the consumption of x_p units of good G_p , $p = \overline{1, n}$: $U(x_1, \dots, x_n) = Ax_1^{\alpha_1} \dots x_n^{\alpha_n}$ where α_p is the elasticity of utility in relation to the good G_p , and A is a positive constant.

The issue of maximizing the utility relative to the restriction: $\sum_{k=1}^n p_k x_k \leq V$ is:

$$\begin{cases} \max U(x_1, \dots, x_n) \\ \sum_{k=1}^n p_k x_k \leq V \\ x_1, \dots, x_n \geq 0 \end{cases}$$

Considering the Lagrangeian:

$$\Phi(x_1, \dots, x_n, \lambda) = U(x_1, \dots, x_n) + \lambda \left(\sum_{k=1}^n p_k x_k - V \right)$$

the maximum condition with restrictions must satisfy:

$$\begin{cases} \frac{\partial \Phi}{\partial x_j} = \frac{\partial U}{\partial x_j} + \lambda p_j = 0, j = \overline{1, n} \\ \frac{\partial \Phi}{\partial \lambda} = \sum_{j=1}^n p_j x_j - V = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha_j A x_1^{\alpha_1} \dots x_j^{\alpha_j-1} \dots x_n^{\alpha_n} + \lambda p_j = 0, j = \overline{1, n} \\ \sum_{j=1}^n p_j x_j - V = 0 \end{cases} \Leftrightarrow$$

$$-\sum_{j=1}^n \frac{\alpha_j A x_1^{\alpha_1} \dots x_j^{\alpha_j-1} \dots x_n^{\alpha_n}}{\lambda} x_j - V = 0 \Leftrightarrow \frac{A x_1^{\alpha_1} \dots x_j^{\alpha_j} \dots x_n^{\alpha_n}}{\lambda} \sum_{j=1}^n \alpha_j + V = 0 \Leftrightarrow$$

¹ Associate Professor, PhD, Danubius University of Galati, Romania, Corresponding author: catalin_angelo_ioan@univ-danubius.ro.

² Senior Lecturer, PhD, Danubius University of Galati, Romania, E-mail: ginaioan@univ-danubius.ro.

$$\lambda = -\frac{Ax_1^{\alpha_1} \dots x_j^{\alpha_j} \dots x_n^{\alpha_n} \sum_{j=1}^n \alpha_j}{V}.$$

By re-replacing, we get the optimal solution to the problem:

$$x_j^* = \frac{\alpha_j V}{p_j \sum_{j=1}^n \alpha_j}, j = \overline{1, n}$$

Let us also consider a producer with a number of K capital units, having the price of p_K and L workers whose hourly wage is w for a working time t . If the elasticity of production in relation to capital and labor are constant, the function of production is: $Q(t, K, L) = CK^\beta L^\gamma$ where β and γ are the elasticities of production in relation to the capital, respectively the labor, C being a constant.

Total cost of production: $CT = p_K K + twL$ leads to a gross profit corresponding to a sales price p :
 $\pi(t, K, L) = pQ(t, K, L) - CT = CpK^\beta L^\gamma - p_K K - twL$.

For a given production Q_0 , the profit maximization condition returns to minimizing the total cost, so to

$$\text{the problem: } \begin{cases} \min(p_K K + twL) \\ CK^\beta L^\gamma = Q_0 \\ K, L \geq 0 \end{cases}.$$

Considering the Lagrangeian:

$$\Phi(x_1, \dots, x_n, \lambda) = p_K K + twL + \lambda(CK^\beta L^\gamma - Q_0)$$

the minimum condition with restrictions must satisfy:

$$\begin{cases} \frac{\partial \Phi}{\partial K} = p_K + \lambda \beta CK^{\beta-1} L^\gamma = 0 \\ \frac{\partial \Phi}{\partial L} = tw + \lambda \gamma CK^\beta L^{\gamma-1} = 0 \\ \frac{\partial \Phi}{\partial \lambda} = CK^\beta L^\gamma - Q_0 = 0 \end{cases} \Leftrightarrow \begin{cases} \lambda = -\frac{p_K}{\beta CK^{\beta-1} L^\gamma} \\ \lambda = -\frac{tw}{\gamma CK^\beta L^{\gamma-1}} \\ CK^\beta L^\gamma - Q_0 = 0 \end{cases} \Leftrightarrow \begin{cases} \frac{p_K}{\beta CK^{\beta-1} L^\gamma} = \frac{tw}{\gamma CK^\beta L^{\gamma-1}} \\ CK^\beta L^\gamma - Q_0 = 0 \end{cases}$$

$$\begin{cases} \frac{p_K}{\beta L} = \frac{tw}{\gamma K} \\ CK^\beta L^\gamma - Q_0 = 0 \end{cases} \Leftrightarrow \begin{cases} L^* = \frac{Q_0}{C} \left(\frac{p_K \gamma}{\beta tw} \right)^{\frac{\beta}{\beta+\gamma}} \\ K^* = \frac{Q_0}{C} \left(\frac{p_K \gamma}{\beta tw} \right)^{-\frac{\gamma}{\beta+\gamma}} \end{cases}, \text{ and the minimum total cost:}$$

$$CT^* = \frac{Q_0}{C} \left(p_K \left(\frac{p_K \gamma}{\beta tw} \right)^{-\frac{\gamma}{\beta+\gamma}} + tw \left(\frac{p_K \gamma}{\beta tw} \right)^{\frac{\beta}{\beta+\gamma}} \right).$$

The maximum profit is:

$$\pi^*(t, K^*, L^*) = pQ_0 - \frac{Q_0}{C} \left(p_K \left(\frac{p_K \gamma}{\beta t w} \right)^{-\frac{\gamma}{\beta+\gamma}} + t w \left(\frac{p_K \gamma}{\beta t w} \right)^{\frac{\beta}{\beta+\gamma}} \right)$$

2. The Model

Suppose there are a number of n firms: F_1, \dots, F_n each having a number of L_i employees, $i = \overline{1, n}$ where we will include, for simplification, the entrepreneur of the firm. Let w_i - the average hourly wage for F_i , t_i - the working time during the analysis period in F_i . We will also assume that F_i produces a single good (of constant elasticity): G_i whose sales price is p_i .

From the total revenue received, each employee pays a tax quota to the state budget χ .

For health insurance, pensions and other services that will later be paid back to employees, they pay a share ξ of the salary received. Let us consider the providers of these services (a single service G_{n+j} , $j = \overline{1, m}$ for each firm) as being the firms F_{n+1}, \dots, F_{n+m} each having L_{n+j} , $j = \overline{1, m}$ employees (including the entrepreneur), with w_{n+j} - the average hourly wage corresponding to the company F_{n+j} and t_{n+j} - the working time worked during the analysis period in F_{n+j} . The service price offered by F_{n+j} will be p_{n+j} , $j = \overline{1, m}$.

Therefore, the tax paid by each employee will be: $T_b = \chi w_i t_i$, $i = \overline{1, n+m}$ and for public services: $T_s = \xi w_i t_i$, $i = \overline{1, n+m}$.

The revenues available to F_i staff are therefore (for each individual employee): $V_i = (1 - \chi - \xi) w_i t_i$, $i = \overline{1, n+m}$.

On the other hand, the amount of salaries received by service providers comes from the share ξ

$$\text{therefore: } \xi \sum_{i=1}^{n+m} w_i t_i L_i = \sum_{j=1}^m w_{n+j} t_{n+j} L_{n+j} \quad \text{or: } \sum_{j=1}^m w_{n+j} t_{n+j} L_{n+j} = \frac{\xi}{1-\xi} \sum_{i=1}^n w_i t_i L_i.$$

The entrepreneur of F_i , $i = \overline{1, n+m}$ will allocate the profits made for investments that will be considered as goods produced by firms. Let us consider the output of F_i as: $Q_i(t, K_i, L_i) = C_i K_i^{\beta_i} L_i^{\gamma_i}$ where β_i and γ_i represent the constantly assumed elasticities of production relative to K_i , respectively L_i , C_i - positive constant. At a price of capital p_{K_i} , the total cost of production in F_i becomes: $CT_i = p_{K_i} K_i + t_i w_i L_i$. Therefore, at a sale price p_i of the G_i asset, F_i 's profit is:

$$\pi_i = p_i Q_i(t, K_i, L_i) - p_{K_i} K_i - t_i w_i L_i = p_i C_i K_i^{\beta_i} L_i^{\gamma_i} - p_{K_i} K_i - t_i w_i L_i$$

The F_i 's firm's entrepreneur income will therefore be just that π_i , $i = \overline{1, n+m}$.

Let considering the set S of social assistants (pensioners, people without income etc.) with a number of M people whose incomes represents a share ρ of the taxes paid by employees (the remainder being allocated to government consumption, public works etc.). Their income will therefore be:

$$\rho(TP_b + TS_b) = \rho \chi \sum_{i=1}^{n+m} w_i t_i L_i \quad \text{and the average income per person: } \frac{\rho \chi}{M} \sum_{i=1}^{n+m} w_i t_i L_i.$$

In the following, we will consider that the utility function of any employee of a production, service or social assistance company will be the same for all consumers within the category (it may be different from company to company – as an example, the utility of books is different for employees of an educational establishment and another for meat producers). In addition, we will assume that all the production of a company will be sold.

Consider the utility functions for an employee of F_i : $\tilde{U}_i(x_{i1}, \dots, x_{i, n+m}) = A_i x_{i1}^{\alpha_{i1}} \dots x_{i, n+m}^{\alpha_{i, n+m}}$, $i = \overline{1, n+m}$ where x_{ij} represents the quantity of good G_j consumed by an employee of F_i and for social assistants: $\tilde{U}_S(x_1, \dots, x_{n+m}) = A x_1^{\alpha_1} \dots x_{n+m}^{\alpha_{n+m}}$ where x_j represents the amount of good G_j consumed by a social assistant.

The utility functions of the entrepreneurs in the investment activity will be: $\tilde{U}_i(y_{i1}, \dots, y_{in}) = B_i y_{i1}^{\delta_{i1}} \dots y_{in}^{\delta_{in}}$, $i = \overline{1, n+m}$ where y_{ij} represents the amount of good G_j consumed by the F_i 's entrepreneur.

Every employee and entrepreneurs want to maximize their utilities in the context of disposable income, so problems arise:

- $$\begin{cases} \max \tilde{U}_i(x_{i1}, \dots, x_{i, n+m}) = A_i x_{i1}^{\alpha_{i1}} \dots x_{i, n+m}^{\alpha_{i, n+m}} \\ \sum_{k=1}^{n+m} p_k x_{ik} \leq (1 - \chi - \xi) w_i t_i \\ x_{i1}, \dots, x_{i, n+m} \geq 0 \end{cases} \quad \text{- for company employees } F_i, i = \overline{1, n+m};$$
- $$\begin{cases} \max \tilde{U}_S(x_1, \dots, x_{n+m}) = A x_1^{\alpha_1} \dots x_{n+m}^{\alpha_{n+m}} \\ \sum_{k=1}^{n+m} p_k x_k \leq \frac{\rho \chi}{M} \sum_{i=1}^{n+m} w_i t_i L_i \\ x_{i1}, \dots, x_{i, n+m} \geq 0 \end{cases} \quad \text{- for social assistants;}$$
- $$\begin{cases} \max \tilde{U}_i(y_{i1}, \dots, y_{in}) = B_i y_{i1}^{\delta_{i1}} \dots y_{in}^{\delta_{in}} \\ \sum_{k=1}^n p_k y_{ik} \leq p_i C_i K_i^{\beta_i} L_i^{\gamma_i} - p_{K_i} K_i - t_i w_i L_i, i = \overline{1, n+m} \\ y_{i1}, \dots, y_{in} \geq 0 \end{cases} \quad \text{- for entrepreneurs.}$$

It follows from the above that the optimum quantities of products are:

- $$x_{ij}^* = \frac{\alpha_{ij} (1 - \chi - \xi) w_i t_i}{p_j \sum_{k=1}^{n+m} \alpha_{ik}}, j = \overline{1, n+m} \quad \text{- for company employees } F_i, i = \overline{1, n+m};$$
- $$x_j^* = \frac{\alpha_j \frac{\rho \chi}{M} \sum_{i=1}^{n+m} w_i t_i L_i}{p_j \sum_{k=1}^{n+m} \alpha_k}, j = \overline{1, n+m} \quad \text{- for social assistants;}$$
- $$y_{ij}^* = \frac{\delta_{ij} p_i C_i K_i^{\beta_i} L_i^{\gamma_i} - p_{K_i} K_i - t_i w_i L_i}{p_j \sum_{k=1}^{n+m} \delta_{ik}}, j = \overline{1, n+m} \quad \text{- for entrepreneurs.}$$

Therefore, the amount of required G_j needed is:

$$Q_j = \sum_{i=1}^{m+n} L_i x_{ij}^* + M \sum_{i=1}^{m+n} x_j^* + \sum_{i=1}^{m+n} y_{ij}^* =$$

$$\frac{1-\chi-\xi}{p_j} \sum_{i=1}^{m+n} L_i \frac{\alpha_{ij} w_i t_i}{\sum_{k=1}^{n+m} \alpha_{ik}} + \frac{\rho \chi \alpha_j (m+n) \sum_{i=1}^{n+m} w_i t_i L_i}{p_j \sum_{k=1}^{n+m} \alpha_k} + \frac{1}{p_j} \sum_{i=1}^{m+n} \frac{\delta_{ij} p_i C_i K_i^{\beta_i} L_i^{\gamma_i} - p_{K_i} K_i - t_i w_i L_i}{\sum_{k=1}^{n+m} \delta_{ik}}, j = \overline{1, n+m}.$$

Returning to the problem of maximizing F_j 's profit for the quantity Q_j we have:

$$\begin{cases} \min (p_{K_j} K_j + t_j w_j L_j) \\ C_j K_j^{\beta_j} L_j^{\gamma_j} = Q_j \\ K_j, L_j \geq 0 \end{cases}, j = \overline{1, n+m}$$

from where:

$$\begin{cases} L_j^* = \frac{Q_j}{C_j} \left(\frac{p_{K_j} \gamma_j}{\beta_j t_j w_j} \right)^{\frac{\beta_j}{\beta_j + \gamma_j}} \\ K_j^* = \frac{Q_j}{C_j} \left(\frac{p_{K_j} \gamma_j}{\beta_j t_j w_j} \right)^{\frac{\gamma_j}{\beta_j + \gamma_j}} \end{cases}$$

Noting for simplicity:

$$s_i = w_i t_i, r_{ij} = \frac{\alpha_{ij} s_i}{\sum_{k=1}^{n+m} \alpha_{ik}}, u_j = \frac{\rho \chi (m+n) \alpha_j}{\sum_{k=1}^{n+m} \alpha_k}, v_{ij} = \frac{\delta_{ij} p_i C_i}{\sum_{k=1}^{n+m} \delta_{ik}}, z_i = \frac{p_{K_i}}{\sum_{k=1}^{n+m} \delta_{ik}}, f_i = \frac{s_i}{\sum_{k=1}^{n+m} \delta_{ik}},$$

$$g_j = \frac{\left(\frac{p_{K_j} \gamma_j}{\beta_j t_j w_j} \right)^{\frac{\beta_j}{\beta_j + \gamma_j}}}{C_j p_j} \Rightarrow \frac{p_{K_j} \gamma_j}{\beta_j t_j w_j} = g_j^{\frac{\beta_j + \gamma_j}{\beta_j}} (C_j p_j)^{\frac{\beta_j + \gamma_j}{\beta_j}} \text{ from where: } \frac{\left(\frac{p_{K_j} \gamma_j}{\beta_j t_j w_j} \right)^{\frac{\gamma_j}{\beta_j + \gamma_j}}}{C_j p_j} = g_j^{\frac{\gamma_j}{\beta_j}} (C_j p_j)^{\frac{\gamma_j}{\beta_j} - 1}$$

we find that:

$$\begin{cases} L_j^* = g_j \left((1-\chi-\xi) \sum_{i=1}^{m+n} r_{ij} L_i^* + u_j \sum_{i=1}^{n+m} s_i L_i^* + \sum_{i=1}^{m+n} (v_{ij} K_i^{*\beta_i} L_i^{*\gamma_i} - z_i K_i^* - f_i L_i^*) \right) \\ K_j^* = g_j^{\frac{\gamma_j}{\beta_j}} (C_j p_j)^{\frac{\gamma_j}{\beta_j} - 1} \left((1-\chi-\xi) \sum_{i=1}^{m+n} r_{ij} L_i^* + u_j \sum_{i=1}^{n+m} s_i L_i^* + \sum_{i=1}^{m+n} (v_{ij} K_i^{*\beta_i} L_i^{*\gamma_i} - z_i K_i^* - f_i L_i^*) \right) \end{cases}$$

or, in other words:

$$\begin{cases} L_j^* = \sum_{i=1}^{m+n} g_j (r_{ij} (1-\chi-\xi) + u_j s_i - f_i) L_i^* + \sum_{i=1}^{m+n} g_j v_{ij} K_i^{*\beta_i} L_i^{*\gamma_i} - \sum_{i=1}^{m+n} g_j z_i K_i^* \\ K_j^* = \sum_{i=1}^{m+n} g_j^{\frac{\gamma_j}{\beta_j}} (C_j p_j)^{\frac{\gamma_j}{\beta_j} - 1} (r_{ij} (1-\chi-\xi) + u_j s_i - f_i) L_i^* + \sum_{i=1}^{m+n} g_j^{\frac{\gamma_j}{\beta_j}} (C_j p_j)^{\frac{\gamma_j}{\beta_j} - 1} v_{ij} K_i^{*\beta_i} L_i^{*\gamma_i} - \sum_{i=1}^{m+n} g_j^{\frac{\gamma_j}{\beta_j}} (C_j p_j)^{\frac{\gamma_j}{\beta_j} - 1} z_i K_i^* \end{cases}$$

Noting again:

- $Y_{1,ij} = g_j (r_{ij}(1 - \chi - \xi) + u_j s_i - f_i)$
- $Y_{2,ij} = g_j v_{ij}$
- $Y_{3,ij} = g_j z_i$
- $Y_{4,ij} = g_j^{\beta_j} (C_j p_j)^{\beta_j - 1} (r_{ij}(1 - \chi - \xi) + u_j s_i - f_i)$
- $Y_{5,ij} = g_j^{\beta_j} (C_j p_j)^{\beta_j - 1} v_{ij}$
- $Y_{6,ij} = g_j^{\beta_j} (C_j p_j)^{\beta_j - 1} z_i$

we obtain:

$$\begin{cases} L_j^* = \sum_{i=1}^{m+n} Y_{1,ij} L_i^* + \sum_{i=1}^{m+n} Y_{2,ij} K_i^{\beta_j} L_i^{*\gamma_i} - \sum_{i=1}^{m+n} Y_{3,ij} K_i^* \\ K_j^* = \sum_{i=1}^{m+n} Y_{4,ij} L_i^* + \sum_{i=1}^{m+n} Y_{5,ij} K_i^{\beta_j} L_i^{*\gamma_i} - \sum_{i=1}^{m+n} Y_{6,ij} K_i^* \end{cases}, j=1, n+m$$

The system solution will provide the optimal number of employees of each firm as well as the required capital.

On the other hand, provided that: $\sum_{j=1}^m w_{n+j} t_{n+j} L_{n+j} = \frac{\xi}{1-\xi} \sum_{i=1}^n w_i t_i L_i$ follows:

$$\sum_{j=1}^m w_{n+j} t_{n+j} L_{n+j}^* = \frac{\xi}{1-\xi} \sum_{i=1}^n w_i t_i L_i^*.$$

By replacing the above optimal solutions, we obtain the link between the two quotas (the only ones that are imposed at government level): χ (tax) and ξ - for health insurance, pensions and other services.

3. References

Ioan, C.A. & Ioan, G. (2011). *n-Microeconomics*. Galați: Zigotto Publishers.

Ioan, C.A. & Ioan, G. (2012). *Methods of mathematical modeling in economics*. Galați: Zigotto Publishers.