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Article

Stable solutions on many-to-one matching models with quota restriction with substitutable preferences for one side of agents

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Stable solutions on many-to-one matching models with quota restriction with substitutable preferences for one side of agents[1]

Abstract: *In this paper a variant to the many-to-one matching model is presented, in which two types of complementary agents and an institution intervene. The latter wants to assign agents to perform certain tasks, each* $\mu_{(d,e)}$ *of which can be done by one agent from one set with many agents from the other. The institution has preferences over the possible matchings and a quota q, which is the maximum number of agents it can hire. In this model, considering substitutable preferences for one of the agent sets, a concept of stability is extended in a natural way and the concept of* q_E*-stability is defined. It is shown, under the institution's responsive preference constraint, that there is an algorithm by means of which the existence of the set of* q_E*-stable matchings is guaranteed.*

Keywords: *Matching; Quota; Stability; Restriction.*

Classificação JEL: C78; C71; D79.

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1 . Introduction

Many-to-one matching models are used to study market problems whose distinctive feature is that the agents involved from the outset are in disjoint sets with different characteristics (e.g. principals and students, firms and workers, etc.). The nature of the problem studied here consists of assigning to an agent from one set many agents from the other. In the case in which to an agent in a set one agent is assigned at the most, the model is called one-to-one matching model. Agents not matched to any agent in the other set are called *singles*.

The "College admissions problem" is the name given by Gale and Shapley (1962) to the simplest of the many-to-one models. They assume that companies have a maximum number of positions to cover (the quota), each firm has a preference relationship in the different groups of workers and each worker has a preference relationship in the whole of the companies. A central theme in formulating a many-to-one model is how to model the preferences of firms as this involves comparison of different groups of workers. The simplest preference of acceptance of the firms is the *responsive* preference. The firm, before any pair of subsets of workers which differ in only one worker, prefers the subset containing the most preferred worker according to their individual preference over each worker. Given a preference profile, an assignment or matching satisfies a specific property of stability if it cannot be blocked, in a sense to be specified below, by any agent or any unassigned pair of agents.

Roth and Sotomayor (1990) studied the most general of many-to- one models, which they called "College admissions problem with substitutable preferences". Each firm has a substitutive preference relationship over different groups of workers, i.e. each firm prefers to hire a worker, even if other workers are no longer available, regardless of their individual preferences over each worker. Kelso and Crawford (1982) were the first to use this property in the most general model many-to-one with money.

A variant to the one-to-one matching model is the *one-to-one matching model with capacity constraint*, presented by Femenia, Marí, Neme and Oviedo (2011). They assume that two sets of complementary agents and one institution intervene and the model consists of assigning each worker on one side of the market to a worker on the other side, such that the pairs of workers hired by the institution are q, at most. That is to say, the institution will have to choose q pairs of workers at most in agreement with its order of preference. The stability property in this model depends on the preferences expressed by the participants and on the the institution's preferences; this is why the property of q-stability is defined. Under the constraint of the institution's *responsive preferences*, the existence of the set of q-stable matchings is guaranteed and it is possible to obtain its characterization. Femenia and Marí (2012) presented an application of these results to the real estate market with the variant that in the matching process the state intervenes. Marí (2012) presented an algorithm, which, starting from an arbitrary matching of the models, converges to a q-stable matching.

In this work, a variant is proposed to the many-to-one model. It is assumed that an institution U wants to assign each agent in a set D to many agents in a set . The institution has preferences over the agents of both sets and a quota q_E, which is the maximum number of agents it can assign. In addition, it is natural to think that there may be more candidates than positions to be filled. For example, a university has members on its staff who are prospective scholarship directors and gives a certain number of grants to be distributed among students in an optimum way.

The stability property in this model depends on the participants' and the institution's preferences. As in many-to-one matching models, the case is considered in which the directors have substitutable preferences.

This work is organized as follows. Section 2 presents a brief review of the theoretical concepts of many-to-one, one-to-one and one-to-one matching with capacity constraints models. It includes the most important definitions and the results that guarantee the existence of stability in these models. Section 3 describes the many-to-one matching model with capacity constraints. Under substitutable preferences for directors the *q*-blocking concept of the one-to-one model with capacity constraint is extended to this new model in a natural way. Finally, in section 4 we give an algorithm by which, starting from an arbitrary matching model, it converges to an matching q_E -stable, which the existence of q_E-stable matching is guaranteed.

2. Preliminaries: Bilateral matching models

2.1. Many-to-one matching models

The many-to-one matching model consists of two disjoint sets $D=\{d_1,...,d_n\}$ and $E=\{e_1,...,e_m\}$. Each agent $e\in E$ has a strict, complete and transitive preference relation P_E over set $D\cup\{\phi\}$, and each agent $d\in D$ has a strict, complete and transitive preference relation P_d over set2^E. These preferences determine that an agent *f* of set $E(oD)$ prefers alternative *a* to alternative *b*, where *a* and *b* are agents of set D (or subsets of set E), if and only if *a* precedes *b* in the list of preferences of *f*. A preference profile is $(n+m)$-uplas preference relations of the agents of set D and the agents of set E and is represented by $P=\left(P_{d_1},...,P_{d_n};P_{e_1},...,P_{e_m}\right)=(P_D,P_E)$. Given a preference P_E, the agents of D preferred by *e* to set ϕ are acceptable for *e*. Similarly, given a preference P_d , the subsets of E preferred by *d* over to set ϕ are acceptable for *d*.

In order to describe the preferences of an agent, we adopt an abbreviated list that includes only the agents or subsets acceptable for it. For example,

$$P_{d_i}=\{e_1,e_3\},\{e_2\},\{e_1\} \qquad\qquad P_{e_j}=d_{1,}d_3$$

Given two disjoint sets D and E and *a* preference profile $\mathbf{P}$, the many-to-one matching model will be denoted by:

$$\bar{M}=(D,E,\mathbf{P}).$$

A solution of the many-to-one matching model is a mapping that assigns a subset of agents of E to each agent of D. Formally,

an assignment or matching is a function

$\mu: D\cup E\to 2^{D\cup E}$ such that, for every $d\in D$ and $e\in E$, it satisfies:

- $\mu(e)\subseteq D$ y # $\mu(e)=1$ or else $\mu(e)=\phi$
- $\mu(d)\in 2^E$
- $\mu(e)=\{d\}$ if and only if $e\in\mu(d)$

Note 2.1 In condition 3, making overuse of language, $\mu(e)=d$ will be used instead of $\mu(e)=\{d\}$.

The set of all the possible matchings in model $\overline{M}$ will be denoted by $\overline{\mathrm{M}}$.

Given a model $\bar{M}=(D,E,\mathbf{P})$ and a matching $\mu\in\overline{M}$, the following subsets of D

and E will be considered, respectively:

$$\mu(E)=\{d\in D:\mu(d)\neq\phi\} \text{ y } \mu(D)=\{e\in E:\mu(e)\neq\phi\}$$

As usual, the cardinality of sets $\mu(E)$ and $\mu(D)$ is denoted by $\#\mu(E)$ and $\#\mu(D)$, respectively. Set $\mu(E)$ is a set of agents of set D, which suggests that $\#\mu(E)$ be denoted as $\#_D\mu$ and, similarly, $\#\mu(D)$ be denoted by $\#_E\mu$; that is to say, $\#_E\mu=\#\mu(D)$ and $\#_D\mu=\#\mu(E)$.

The subset of E most preferred by *d* with respect to the preference P_d is called *election of agents* set of E with respect to the preferences of agent *d* and denoted by $Ch(E', P_d)$. Formally,

$$Ch(E',P_d)=\max_{P_d}\{E''\subseteq E':E''P_dE'\}$$

A matching $\mu\in\overline{M}$ is **individually rational** if it is not blocked by any agent.

A matching $\mu\in\overline{M}$ is **blocked by a pair** *(d, e)* if $e\notin\mu(d)$, $dP_e\mu(e)$ and $e\in Ch(\mu(d)\cup\{e\},P_d)$.

A matching $\mu\in\overline{M}$ is **stable** if it is not blocked by an agent or by a pair of agents.

Given a many-to-one matching model $\overline{M}=(D,E,\mathbf{P})$ the set of stable matchings in $\overline{M}$ is denoted by $S(\overline{M})$.

A matching μ is said to be one-to-one if each agent *d* is assigned to an agent e, i.e. Condition 2 of Definition 1.1 is replaced by: $\mu(d)\subseteq E$ and $\#\mu(d)=1$ or else $\mu(d)=\varphi$ The model in which every matching is one-to-one is known in the literature as the *marriage model or one-to-one matching model*. Such a model is denoted by $M=(D,E,\mathbf{P})$, $\mathbf{P}=(P_d,P_e)$, P_d and P_E being preferences over sets $E\cup\{\varphi\}$ and $D\cup\{\varphi\}$, respectively. *M* is the set of all possible matchings in the model and *S(M)*, the set of stable matchings.

Theorem 2.1(*Gale and Shapley*, 1962) *If $M=(D,E,\mathbf{P})$ is a one-to-one matching model, then $S(M)\neq\varphi$.*

It is known that in the many-to-one model, set $S(\overline{M})$ may be empty. (See example 2.7 in *Two-sided matching: a study in game-theoretic and analysis* [11].) Such a result is the reason why the literature has focused on the constraint of the preferences of agents *d*.

For each agent *d* there is a positive integer called the quota of the agent *d*.

Consideremos el modelo muchos a uno $M=(D,E,\mathbf{P})$ mas general, en el que los

Let us consider the most general many-to-one model $M=(D,E,\mathbf{P})$, in which the agents $d\in D$ consider each agent $e\in E$ one another's substitute.

Such a restriction of substitutability was introduced by Kelso and Crawford (1982) with the notion of substitutable preference.

Definition 2.1 The preference relation P_d is substitutable if for all $E'\subseteq E$, $e,e'\in E$ $(e\neq e')$, if $e\in Ch(E',P_d)$ then $e\in Ch(E'\setminus\{e'\},P_d)$.

The preference of stable matchings in the many-to-one model with substitutable preferences is guaranteed.

Theorem 2.2 (*Roth and Sotomayor*, 1990) *If* $\overline{M}=(D,E,\mathbf{P})$ is a *many-to-one matching model with q_d - substitutable preferences, then $S(\overline{M})\neq\varphi$.*

From now onwards, in order to refer to the many-to-one model $\overline{M}=(D,E,\mathbf{P})$ with q_d - substitutable preferences, either of these notations is possible: $\overline{M}=(D,E,\mathbf{P},q_d)$ or $\overline{M}=(D,E,\mathbf{P})$, with D - substitutable preferences.

If for for each $d \in D$, $q_d = 1$, the many-to-one model $\bar{M} = (D, E, \mathbf{P}, q_d)$ is reduced to the one-to-one $M = (D, E, \mathbf{P})$ and, for all $\mu \in \overline{M}$.

Given a many-to-one matching model $\bar{M} = (D, E, \mathbf{P})$, $d \in D$ and a matching μ of the model, we define the set of agents of *E* that prefer *their matching in the* μ and we symbolize it with $E_{d,\mu}$. Formally

$$E_{d,\mu} = \{ e \in E : d \, P_e \, \mu(e) \}$$

Definition 2.2 A matching μ of the many-to-one matching model $\bar{M} = (D, E, \mathbf{P})$ is *quasi-stable for d*, $d \in D$, if μ is *q*-individually rational and for all $d \in D$, $S \subseteq E_{d,\mu}$, $S \neq \varphi$ and $\mu(d) \subseteq Ch(\mu(d) \cup S, q_d, P_d)$.

Definition 2.3 A matching μ of the many-to-one matching model $\bar{M} = (D, E, \mathbf{P})$ is *quasi-stable for e*, $e \in E$, if μ is *q*-individually rational and for all *(d , e)* that blocks μ, $\mu(e) = \varphi$.

We symbolize with $QDS(\bar{M})$ a quasi-stable matching for $d \in D$ and with $QES(\bar{M})$, a quasi-stable matchings for $e \in E$.

The following proposition shows that by applying the *ASO* to a quasi-stable matching for the *d's* in the many-to-one $\overline{M}$ matching model, with D-substitutable preferences, a stable matching of the $\overline{M}$ is obtained.

Proposition 2.1 (Cantala, 2004) Let the many-to-one matching model $\bar{M} = (D, E, \mathbf{P})$ with D-substitutable preferences. If $\mu \in QDS(\bar{M})$, then $SO(\mu) \in S(\bar{M})$.

2.2 One-to-one matching model with capacity restriction

Femenia, Marí, Neme and Oviedo (2011) presented a variant to the one-to-one matching models in which two sets of complementary agents and an institution are involved. The institution wants to assign agents to do certain tasks which can be carried out by a pair of complementary agents. It has preferences over each of the pairs of agents it can assign. Many times, the institution has a quota q, which is the maximum number of pairs of agents it can assign.

It must be noted that, even though in this model two sets of workers and an institution are involved, this model is not equivalent to the trilateral matching model introduced by Alkan (1986), in which there is no stability.

This model consists of two finite and disjoint sets of agents denoted by $D = \{d_1, ..., d_n\}$ and $E = \{e_1, ..., e_m\}$, respectively.

Each worker $d \in D$ has a strict preference relation[4] P_d over the set of agents 2^E and each worker $e \in E$ has a strict preference relation P_E over the set of agents $D \cup \{\varphi\}$.

Preference profiles are *(n+m)*-tuples of preference relations represented by $\mathbf{P} = \left(P_{d_1}, ..., P_{d_n}; P_{e_1}, ..., P_{e_m}\right) = (P_D, P_E)$, and an institution denoted by U. Institution U has a binary relation R_U over the set of all possible matchings M, the empty matching included. Let P_U and I_U denote the strict and indifferent preference relations induced by R_U, respectively. Suppose now that the institution can assign a maximum number of positions - quota $q \leq \min\{n, m\}$ - to be filled; then, only the matchings whose cardinality is smaller or equal to *q* may be acceptable. The institution may choose some matchings of M according to its preference P_U and their quota restriction *q*. We denote $M_q = \{\mu \in M : \# \mu \leq q\}$.

This new matching marker is denoted by $M_U^q = (M, R_U, q)$.

A matching μ is *acceptable* for institution U according to their preferences if $\mu \in$

[4] A preference is a binary, reflexive, antisymmetric, transitive and complete relation.

M_q and $\mu R_U \mu^{\varphi}$, in which μ^{φ} is the matching such that $\mu^{\varphi}(x)=\varphi$, for every $x \in D \cup E$.

Given M and a quota $q \leq \min\{n,m\}$, the institution can only accept assignments of M which are most preferred to the empty matching according to its preference P_U, and its cardinal is not larger than the allowed number of positions $\#\mu \leq q$. A matching is acceptable if the partner assigned is preferred to the empty set. Formally,

Definition 2.4 Given a model M_U^q, an assignment μ is *q-individually rational* if $\#\mu \leq q$ $\mu P_U \varphi$ and for every $f \in D \cup E$ such that $\mu(f) P_f \varphi$ is verified.

Given an assignment $\mu \in$M and a pair of workers $(d,e) \in D \times E$, $\mu_{(d,e)}$ is defined as follows: $\mu_{(d,e)}(f) = \begin{cases} \mu(f) & if \quad f \notin \{d,e,\mu(d),\mu(e)\} \\ d & if \qquad f=e \\ \varphi & \qquad otherwise. \end{cases}$

Notice that if $\mu(d)=e$, then $\mu_{(d,e)}=\mu$.

Note 2.2 The matching $\mu_{(d,e)}$ may not be individually rational. Let us consider a matching μ such that $\#\mu = q$ and let *(d,e)* such that, if $\mu(d)=\varphi=\mu(e)$, then $\#\mu_{(d,e)} > q$ and $\mu_{(d,e)}$ is not *q*-individually rational.

Usually, in standard models, *(d,e)* is a blocking pair if these agents are not assigned to each other and if they each other to their current partners. Note that in our model, we may have a blocking pair *(d,e)* such that the matching that the blocking pair satisfies is not acceptable for institution U. Then , we will consider two types of blocking pairs for μ. One type is that which occurs when the assignment μ is blocked by a couple of agents in the institution, already assigned by the matching, and the other is the type in which the assignment is blocked by a pair of agents, one of whom at least is single. In this case, the assignment obtained, which satisfies the blocking pair, is preferred by the institution to the assignment μ. Formally:

Definition 2.5 A matching μ is *q-blocked* by a pair of workers *(d,e)* if

1. $eP_d\mu(d), dP_e\mu(e)$, and

2. either

(a) $\mu(d) \in E$ and $\mu(e) \in D$, or

(b) $\mu_{(d,e)}$ is *q*-individually rational and $\mu_{(d,e)} R_U \mu$,

Definition 2.6 *A matching μ is q-stable if it is q-individually rational and is not q-blocked by any pair of workers.*

Given a matching market $M_U^q=(M;R_U,q)$, $S(M_U^q)$ denotes the set of *q*-stable matchings. Notice that in Femenia, Marí, Neme and Oviedo (2011) it was proved that, under the restriction of the institution's responsive preferences, the set of *q*-stable matchings is non-empty, i.e. $S(M_U^q) \neq \varphi$. They also obtained a characterization of this set as: $S(M_U^q)=T_q(M) \cup T_{<q}(M)$.

Note 2.3 The definition of the institution´s responsive preferences and of sets $T_q(M)$ and $T_{<q}(M)$ are given in detail en Appendix.

3. The Model

A variant to the many-to-one matching model will be considered now in which two sets of complementary agents and an institution are involved. The institution wants to assign agents to do certain tasks each of which can be performed by one agent of a set of many agents of the complementary set. The institution has preferences over each of the pairs of agents it can assign. Unlike the one-to-one matching model with capacity restriction this model matches an agent from set D with many agents from set E, and the institution has a quota which is the maximum number of agents from E it can assign. Since the institution's quota limitation is given over agents from set E, it is symbolized with q_E. Each $d \in D$ has a maximum number of agents from E with which it may be designated which will be indicated with q_d.

We assume that $q_E \leq \min\left\{ \# E , \sum_{d \in D} q_d \right\}$.

This new matching model is called many-to-one matching model with capacity restriction and denoted by $\bar{M}_U^{q_E} = \left(\bar{M}, R_U, q_E\right)$.

The set of all matchings in this model is symbolized $\bar{M}_{q_E}$ with, i.e., $\bar{M}_{q_E} = \left\{ \mu \in \bar{M} : \#_E \, \mu \leq q_E \right\}$, where $\#_E \, \mu = \#\{e \in E : \mu(e) \neq \varphi\} = \sum_{e \in E} \# \mu(e)$.

The notion of the q_E-individually rational matching of the one-to-one model with capacity restriction is extended naturally to the many-to-one model with capacity restriction as follows.

Definition 3.1 Given a model $\bar{M}_U^{q_E}$, a matching $\mu \in \bar{M}_{q_E}$ is q_E-individually rational if for all $e \in E$, $\mu(e) \, P_d \, \varphi$, for all $d \in D$, $\mu(d) = Ch(\mu(d), q_d, P_d)$ and $\mu \, R_U \, \mu^{\varphi}$.

Consider the model $\bar{M}_U^{q_E} = \left(\bar{M}, R_U, q_E\right)$ with D-responsive preferences. It is noted that if for every $d \in D$, $q_d = 1$, the model is reduced to the one-to-one model with capacity restriction. The objective is to extend naturally to this model the definition of q-blocking of the one-to-one model with capacity restriction.

Definition 3.2 A matching μ is q_E-*blocked* by a pair of workers *(d,e)* if

1. $e \notin \mu(d)$, $d \, P_e \, \mu(e)$, $e \in Ch(\mu(d) \cup \{e\}, q_d, P_d)$, and

2. either

(a) $\mu(d) \neq \varphi$ and $\mu(e) \neq \varphi$, or

(b) there exists $E' = \mu(d) \cup \{e\}$, $\mu_{(d,e)}^{E'}$ is *q*-individually rational and $\mu_{(d,e)}^{E'} R_U \, \mu$,

$$\mu_{(d,e)}^{E'}(f) = \begin{cases} Ch\left(E', q_d, P_d\right) & if & f = d \\ d & if & f = e \\ \mu(e')\backslash\{\mathrm{d}\} & \mathrm{if} & \mathrm{f} = \mathrm{e}' \text{ and } \mathrm{e}' \in \left(E' \backslash Ch\left(E', q_d, P_d\right)\right) \\ \mu(f)\backslash\{\mathrm{e}\} & \mathrm{if} & f = \mu(e) \\ \mu(f) & & otherwise \end{cases}$$

Definition 3.3 *A matching* μ *is* q_E*- stable if it is* q_E*-individually rational and is not* q_E*-blocked by any pair of agents.*

Given a matching market $\bar{M}_U^{q_E} = \left(\bar{M}, R_U, q_E\right)$, $S\left(\bar{M}_U^{q_E}\right)$ denotes the set of q_E-stable matchings.

4. Existence of stable solution

Some valid results, necessary for subsequent sections, are presented.

4.1 An algorithm leading to stable matching in the many-to-one matching models with D-substitutable preferences

Cantala (2004) presents an algorithm in the many-to-one matching model with q_d -substitutable, with which he affirms the stabilization of any quasi-stable matching, a notion that will be defined later. Such an algorithm is called an offer set algorithm (OSA).

The matching obtained by applying the OSA to μ is symbolized with $SO(\mu)$.

The OSA is described below. For each iteration *i*, we consider the set of agents of E that want to join the agent $d \in D$. A_d^{i-1} is the set of agents of E that have neither rejected an offer from *d* nor been assigned with him until this iteration *i* and belongs to the subset of E acceptable for *d*. Within this set, each agent $d \in D$ makes an offer to the agents of E in S_d^{i-1} (subset of A_d^{i-1} that is not in $\mu^{i-1}(d)$, that is the $e \in A_d^{i-1}$ that are not assigned in iteration *i*- 1 with *d*) which are more preferred with respect to the preferences of *d*, of less cardinality than $q_d - \# \mu^{i-1}(d)$. *E's* agents accept the best proposed offer between the agent assigned in the matching and the *D* agents that were offered in this iteration. Given the new matching, D's agents make new proposals in a new iteration. If there is no *D* agent who wants to bid, the dynamic stops.

In order to formalize the steps of the OSA, for each $d \in D$, we define the following sets

$A_d^i = \left\{ e \in E : e \notin \mu^i(d) \text{ and there is } E' \subseteq E, \# E' < q_d \text{ and } E' \cup \{e\} \, P_d \, E' \right\}$

$S_d^i \subseteq A_d^i$ such that it verifies $\# S_d^i + \# \mu^i(d) \leq q_d$ and does not exist $\overline{S} \subseteq A_d^i$ such that $\# \overline{S} + \# \mu^i(d) \leq q_d$ and $\mu^i(d) \cup \overline{S} \, P_d \, \mu^i(d) \cup S_d^i$.

For each $e \in E$, we define the following sets

$T_e^i = \left\{ d \in D : d = \mu^i(d) \text{ or } e \in S_d^i \right\}$.

We describe the steps of the OSA below.

Initiation

(a) $\mu^0 = \mu'$.

(b)For all $d \in D$, let A_d^0.

Iteration i

(1)For all $d \in D$, let S_d^{i-1}.

(2)If for all $d \in D$, $S_d^{i-1} = \varphi$ the algorithm stops; otherwise.

(3)If $S_d^{i-1} = \varphi$, for some $d \in D$, *d* makes offers to $e \in S_d^{i-1}$.

(4)For each E that received offers in step (3), let T_e^{i-1} and define the matching $\mu^i(e) = máx \, T_e^{i-1}$ and for all $e' \in E$ who did not receive offers in step (2), $\mu^i(e) = \mu^{i-1}(e)$.

(5)Let $\mathrm{A}_{\mathrm{d}}^{\mathrm{i}} = \mathrm{A}_{\mathrm{d}}^{\mathrm{i-1}} \setminus S_d^{i-1}$.

(6)Go to step (1).

In the following example we show the steps of the OSA algorithm.

Example 4.1 Let $\overline{M} = (D, E, \mathbf{P})$ be *a many-to-one matching model with* $D = \{d_1, d_2\}$, $E = \{e_1, e_2, e_3\}$ and let the following (2.2)-substitutable preferences be given by:

$P_{d_1} = \{e_1, e_2\}, \{e_2\}, \{e_1\}, \{e_3\}$ $\quad$ $P_{e_1} = P_{e_2} = d_1, d_2$

$P_{d_2} = \{e_3\}, \{e_1, e_2\}, \{e_1\}, \{e_2\}$ $\quad$ $P_{e_3} = d_2, d_1$.

Consider the matching $\mu^0 = \begin{pmatrix} d_1 & d_2 & \varphi \\ e_3 & e_2 & e_1 \end{pmatrix}$,

For d_1 is $A_{d_1}^0 = \{e_1, e_2\}$ and for d_2 is $A_{d_2}^0 = \{e_1, e_3\}$.

Iteration 1

For d_1 and d_2, $S_{d_1}^0 = \{e_2\}$ and $S_{d_2}^0 = \{e_3\}$.

As $T_{e_2}^0 = \{d_1, d_2\}$ and $\underset{e_2}{máx} \, T_{e_2}^0 = \{d_1\}$, then $\mu^1(e_2) = d_1$.

Further $T_{e_3}^0 = \{d_1, d_2\}$ and $\underset{e_3}{máx} \, T_{e_3}^0 = \{d_2\}$, then $\mu^1(e_3) = d_2$.

As $e_1 \notin S_{d_1}^0$ and $e_1 \notin S_{d_2}^0$, $\mu^1(e_1) = \mu^0(e_1)$.

The matching obtained in this iteration is:

$\mu^1 = \begin{pmatrix} d_1 & d_2 & \varphi \\ e_2 & e_3 & e_1 \end{pmatrix},$

Iteration 2

For d_1 and d_2, $A^1_{d_1} = S^1_{d_1} = A^1_{d_2} = S^1_{d_2} = \{e_1\}$.

As $T^1_{e_1} = \{d_1, d_2\}$ and $\underset{e_1}{máx}\, T^1_{e_1} = \{d_1\}$, then $\mu^2(e_2) = d_1$.

The matching obtained in this *Iteration* 2 is

$\mu^1 = \begin{pmatrix} d_1 & d_2 \\ e_1, e_2 & e_3 \end{pmatrix}$

In Iteration 3, $A^2_{d_1} = A^2_{d_2} = \varphi$, and $SO(\mu^0) = \mu^2$.

The following lemma presents a link between the cardinality of the matching μ and $SO(\mu)$.

Lemma 4.1 Let $\bar{M} = (D, E, \mathbf{P})$ *be a many-to-one matching model with* D-substitutable preferences and μ a matching such that $\#_E\, \mu = q$ and $\mu' = SO(\mu)$, then $\#_E\, \mu' \geq q$.

Proof. Let the model $\overline{M}$ and let μ be a matching such that $\#_E\, \mu = q$.

Let the matchings μ^{i-1} and μ^i, obtained in stages *i-1 and i* of the ASO, respectively.

We will prove that $\#_E\, \mu^{i-1} \leq \#_E\, \mu^i$, i.e. , $\#\mu^{i-1}(D) \leq \#\mu^i(D)$.

Suppose that for some $d \in D$, $A^i_d = \varphi$, the algorithm stops and $\mu^{i-1} = \mu^i$, then $\#\mu^{i-1}(D) = \#\mu^i(D)$.

Suppose that for some $d \in D$, $A^i_d \neq \varphi$. Let $e \in \mu^{i-1}(D)$, then $\mu^{i-1}(e) = d$ in the stage *i* of the ASO, the following can occur:

Caso 1 $T^i_e = \varphi$.

If $T^i_e = \varphi$, by step (4) of the ASO $\mu^i(e) = \mu^{i-1}(e) = \bar{d}$, then $e \in \mu^i(D)$. Therefore, $\mu^{i-1}(D) \subseteq \mu^i(D)$, and it is verified that $\#\mu^{i-1}(D) = \#\mu^i(D)$.

Caso 2 $T^i_e \neq \varphi$.

If $T^i_e \neq \varphi$, there exists $d' \in D$ such that $\underset{e}{máx}\, T^i_e = \{d'\}$ and by step (4) of the ASO, $\mu^i(e) = d'$, then $e \in \mu^i(D)$. Therefore, $\mu^{i-1}(D) \subseteq \mu^i(D)$, and $\#\mu^{i-1}(D) = \#\mu^i(D)$ is verified.

4.2 An algorithm to calculate a q_E-stable matching

Given model $\overline{M}$, with D-substitutable preferences and taking into account Theorem 2.2 , which ensures that $S(\bar{M}) \neq \varphi$, for an arbitrary matching of $S(\bar{M}) \neq \varphi$ we will give an algorithm that will allow us to find, with probability one, a q_E-stable matchings in model $\bar{M}^{q_E}_U$ with D-substitutable and U-responsive preferences.

The desired matchings in model $\bar{M}^{q_E}_U$, are those whose cardinality does not exceed q_E, this will allow us to consider the set: $E^t = E \setminus \{ e \in E : e_t \succ_E e \} \subseteq E$ and define the following matching:

Definition 4.1 Let μ be a matching of model $\overline{M}$ and $\succ_E$ the individual preferences of the institution on $E \cup \varphi$. Matching μ is a **truncation** of μ with respect to e_t, if

$$\mu_{e_t}(e) = \begin{cases} \mu(e) & if \quad e \in E^t \\ \varphi & otherwise \end{cases}$$

Observation 4.1

1) From now on we consider μ_t a truncation of the matching μ with respect to some $e_t \in E$ and to refer to it we simply write that μ_{e_t} is a truncation of matching μ.

2) If $\#_E\ \mu_{e_t} = q_E$, we say that μ_{e_t} is a q_E-truncation of the matching μ.

In the following example we show how to find a matching truncation.

Exemple 4.2 Example 4.2 Let $\bar{M}_U^s$ with $D = \{d_1, d_2, d_3, d_4\}$, $E = \{e_1, e_2, e_3, e_4, e_5\}$ and let the following institution preferences: $e_1 \succ_E e_4 \succ_E e_5 \succ_E e_2 \succ_E e_3$.

Consider the matching $\mu = \begin{pmatrix} d_1 & d_2 & d_3 & d_4 \\ e_1\ e_2 & e_3 & e_4\ e_5 & \varphi \end{pmatrix}$.

Matching $\mu_{e_5} = \begin{pmatrix} d_1 & d_2 & d_3 & d_4 & \varphi & \varphi \\ e_1 & \varphi & e_4\ e_5 & \varphi & e_2 & e_3 \end{pmatrix}$, is a 3-truncation of μ with respect to e_5.

Observation 4.2 If μ_{e_t} is a truncation of μ it is verified that μ, then $\mu_{e_t}(D) \subset \mu(D)$ and $\mu_{e_t}(E) \subset \mu(E)$.

Lemma 4.2 Let $\overline{M}$ be the many-to-one matching model with D-substitutable preferences, and $\mu \in S(\overline{M})$. If μ_{e_t} is a truncation from μ, then $\mu_{e_t} \in QDS(\overline{M}')$.

Proof. Let $\mu \in S(\overline{M})$ and μ_{e_t} a truncation from μ. If $\mu_{e_t} \in S(\overline{M})$, then $\mu_{e_t} \in QDS(\overline{M}')$ and the lemma is demonstrated.

Suppose that $\mu_{e_t} \notin QDS(\overline{M}')$, then there is *(d ,e)* that blocks μ_{e_t} in $\overline{M}'$, i.e.

$$e \notin \mu_{e_t}(d), \quad d\, P_e\, \mu_{e_t}(e), \quad e \in Ch\left(\mu_{e_t}(d) \cup \{e\}, q_d, P_d\right) \tag{1}$$

Furthermore, by definition of the matching μ_{e_t}, $e \in E'$ and $\mu_{e_t}(e) = \mu(e)$.

By Observation 4. 1, and definition of the matching μ_{e_t}, the two following cases can occur:

Case 1: $\mu_{e_t}(d) = \mu(d)$.

By replacing in affirmation (1), *d* for *e*, $\mu_{e_t}(e) = \mu(e)$ and $\mu_{e_t}(d) = \mu(d)$, we obtain that *(d ,e)* blocks μ and contradicts that $\mu \in S(\overline{M})$.

Case 2: $\mu_{e_t}(d) \subset \mu(d)$.

As $\mu_{e_t}(d) \subset \mu(d)$, there exists $S = \{e_1, \dots, e_k\} \subseteq E$ such that

$$\mu(d) = \mu_{e_t}(d) \dot{\cup} S \tag{2}$$

Since $\mu_{e_t}(e) = \mu(e)$ and (2) being verified, we obtain that $d\, P_e\, \mu_{e_t}(e)$, then $e \in E_{d,\mu} = \{ e \in E : d\, P_e\, \mu_{e_t}(e) \}$.

Let us prove that $\mu_{e_t}(d) \subseteq Ch\left(\mu_{e_t}(d) \cup \{e\}, q_d, P_d\right)$.

Let $\overline{e} \in \mu_{e_t}(d)$, for each $e_i \in S$, by (2), $\overline{e} \neq e_i$ and

$$\overline{e}, e_i \in \mu_{e_t}(d) \dot{\cup} S \tag{3}$$

Since $\mu \in S(\overline{M})$, *(d ,e)* does not block μ, by (1) and $\mu_{e_t}(e) = \mu(e)$, $d\, P_e\, \mu(e)$ is verified , then $e \notin Ch(\mu(d) \cup \{e\}, q_d, P_d)$. We have now that $Ch(\mu(d) \cup \{e\}, q_d, P_d) = \mu(d)$, and by replacing (2),

$$Ch\left(\mu_{e_t}(d) \dot{\cup} S \cup \{e\}, q_d, P_d\right) = \mu_{e_t}(d) \dot{\cup} S \tag{4}$$

From (3),

$$\overline{e} \in Ch\left(\mu_{e_t}(d) \dot{\cup} S \cup \{e\}, q_d, P_d \right) \tag{5}$$

For each $\overline{e} \neq e_i$, with D-substitutable preference,

$\overline{e} \in Ch\left(\left(\mu_{e_t}(d) \dot{\cup} S \cup \{e\}\right) \setminus \{e_i\}, q_d, P_d\right)$, then $\overline{e} \in Ch\left(\mu_{e_t}(d) \cup \{e\}, q_d, P_d\right)$. Therefore, $\mu_{e_t}(d) \subseteq Ch\left(\mu_{e_t}(d) \cup \{e\}, q_d, P_d\right)$ and $\mu_{e_t} \in QDS(\overline{M}^t)$.

Given μ a stable matching in the many-to-one matching model with D-substitutable preferences, Lemma 4.1, Lemma 4.2 and Proposition 2.1, we are allowed to describe a procedure whereby we can find a new assignment that is q_E-stable in the many-to-one matching model with capacity constraint, with U-responsive preferences, as follows:

If $\#_E\, \mu_{e_t} \leq q_E$, the process ends.

If $\#_E\, \mu_{e_t} > q_E$ we choose t_1, such that $E^{t_1} \subset E$ and a q_E-truncation of the matching μ, which we will call $\mu_{e_{t_1}}$. If this truncation is stable on $\overline{M}^t$, we find the desired assignment.

If $\mu_{e_{t_1}}$ is not stable, we apply the ASO to $\mu_{e_{t_1}}$ in model $\overline{M}^{(m,t_1)} = \overline{M}^{t_1}$, which will be indicated by $SO(\mu_{e_{t_1}}) = \mu^1$. Let us observe that $E^{t_1} \subset E$. We repeat the previous process and we get μ^2 stable matching in $\overline{M}^{(m,t_2)}$ such that $t_2 < t_1$. Consequently, with the iterated application of this process we obtain a q_E-stable matching. Note that in each iteration the procedure is applied to a subset of E strictly included in the previous one.

The following describes the steps to follow; we will indicate it with ASO_{q_E}.

1.Let $\mu \in S(\overline{M})$.

2.If $\#_E\, \mu \leq q_E$ the process ends.

3.If $\#_E\, \mu > q_E$, then let t_1 such that $E^{t_1} \subset E$ and a q_E-truncation of the matching μ such that $\#_E\, \mu_{e_{t_1}} = q_E$.

4.If $\mu_{e_{t_1}} \in S(\overline{M}^{t_1})$, then $\mu_{e_{t_1}} \in S(\overline{M}_U^{q_E})$ and the process ends.

5. $\mu_{e_{t_1}} \notin S(\overline{M}_U^{q_E})$, be $SO(\mu_{e_{t_1}}) = \mu^1 \in S(\overline{M}^{t_1})$. As $\#_E\, \mu^1 \geq q_E$:

1.1 If $\#_E\, \mu^1 = q_E$ go to step 3.

1.2 If $\#_E\, \mu^1 > q_E$ go to step 2.

We symbolize with $SO_{q_E}(\mu)$ the matching obtained by applying the ASO_{q_E} to μ.

The existence of set $S(\overline{M})$, with D-substitutable preferences (Theorem 2), and the algorithm described above allows us to ensure the existence of the set $S(\overline{M}_U^{q_E})$, as we enunciate it in the following theorem:

Proposition 4.1 Let $\overline{M}$ be the many-to-one matching model with D-substitutable preferences, let $\mu \in S(\overline{M})$ and let $\overline{M}_U^{q_E}$ be the many-to-one matching model with capacity restriction, then there is a finite sequence of matchings $\mu_1, \mu_2, \cdots, \mu_k$, obtained by the algorithm given above , such that $\mu_k \in S(\overline{M}_U^{q_E})$.

Proof Let $\overline{M}$ and $\mu \in S(\overline{M})$. We will consider the following cases:

Case 1: $\#_E\, \mu \leq q_E$.

The theorem is trivially true.

Case 2: $\#_E\, \mu > q_E$.

If $\#_E\, \mu > q_E$, we perform a q_E-truncation of μ, that is let $E^{t_1} = E \setminus \{ e \in E : e_{t_1} \succ_E e \}$ and $\mu_{e_{t_1}}$ a truncation of μ, such that $\#_E\, \mu_{e_{t_1}} \leq q_E$.

Consider the reduced model $\overline{M}^{(m,t_1)} = \overline{M}^{t_1}$. If $\mu_{e_{t_1}} \in S(\overline{M}^{t_1})$, the theorem remains demonstrated. Otherwise, as $\mu_{e_{t_1}}$ is a q_E-truncation of μ and $\mu \in S(\overline{M})$, then by Lemma

4. 2, $\mu_{e_{n}} \in QDS(\bar{M}^{t_1})$.

Let $SO_{q_E}(\mu_{e_{n}})=\mu^1$ then, by Proposition 2.1.

By Lemma 4.1, $\#_E\, \mu^1 > \#_E\, \mu_{e_{n}}$.

As $\#_E\, \mu_{e_{n}} = q_E$, the following two cases can occur:

Case 2.1: $\#_E\, \mu^1 = q_E$.

If $\#_E\, \mu^1 = q_E$, as $\mu^1 \in S(\bar{M}^{t_1})$, then $\mu^1 \in S(\bar{M}_U^{q_E})$ and the theorem is proved.

Case 2.2: $\#_E\, \mu^1 > q_E$.

If $\#_E\, \mu^1 > q_E$, considering $E^{t_1} \subset E^{t_1} \subset E$, we perform a q_E-truncation of μ^1 q-truncation , and follow the same procedure as that performed in case 2.

Let us observe that in each iteration a model is obtained such that the set of agents of E considered is strictly included in the set obtained above, which guarantees that in at most $m - q_E$ steps we will obtain the desired stable matching.

Theorem 4.1 $\bar{M}_U^{q_E}$ the model of many-to-one matchings with capacity restriction, where R_U is responsive and D-substitutable preferences, then $S(\bar{M}_U^{q_E}) \neq \varphi$.

5. Comments and Conclusions

Among the different examples of many-to-one markets and matching problems linked to them are those of institutions subsidized by the state and the employees to be hired. The characteristics of this market generate problems that affect mainly those groups of competent low-income workers. Because of this, it is necessary to design long-term integral strategies to produce equitable solutions for both the institutions and the workers; for this purpose, state actions should focus exclusively on sections qualified for certain tasks, which currently do not have access to work in institutions. Now, the state budget is limited and, as a consequence, it is often not possible to carry out all the possible matchings between institutions and workers that ask for that benefit. In other words, this model consists of a set of institutions, a set of workers and the state. Each institution has preferences for potential workers, each potential worker has preferences for potential companies, and the state has a priority over the possible "company-workers" pairs that can be agreed on.

This new model solves the problem in which the companies and the workers match with each other in such a way that they satisfy a stability property that depends on the preferences expressed by the participants and the state's preference. This property consists of no worker (company) having to work (hire) for an institution (workers) he cannot, or he does not want to work for. In addition, there is no "company-workers" pair preferring to reach an agreement different from the one assigned by the state; finally, all the "company-workers" pairs which reach an agreement are accepted by the state and do not exceed the budget it has. When this does not happen, the "company-workers" pair is said to block the matching. Besides, a solution is presented to problems such as the state's budget cuts or the assignment of money to other services for different reasons - global financial crisis, Covid 19 pandemic, etc. In this context, the assignments granted have to be interrupted and the new ones have to satisfy the stability property.

This work guarantees that the state's actions to give solutions in matters of work in accordance with workers' qualifications, with a limited budget, can be carried out with success for both the state and those who have access to the benefit. In other words, it is feasible to find solutions immune to the possibility of companies and workers not agreeing on the benefit distribution, or of the state not making a good distribution of the budget assigned. Even if the state's budget has to be cut, solutions as well as means to achieve them can be found.

The difference from the jobs listed is that I now work in a many-to-one matching with substitutable preferences for one side of agents. The previous ones are from the one-to-one matching and with responsive preferences for the two sets of agents. There are also many-to-one results with other types of agent preferences. Also, the many-to-many matching models with quota restriction are being studied.

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Appendix

The restriction of M

From now on, we will denote $F \in \{D,E\}$ and $F^c \in \{D,E\}$ such that $\{F, F^c\} = \{D,E\}$, and $f \in F$ will denote a generic worker.

Given $F' \subseteq F$, the restriction of P_F to F' will be denoted by $P_{|F'}$. Given $M = (F, F^C, \mathbf{P})$, the restriction of M to F' will be denote by $M_{F'} = (F', F^C, P_{|F'}, P_{F^C})$. For the sake of simplicity we denote $M_{F'} = (F', F^C, \mathbf{P})$, where $\mathbf{P} = (P_{|F'}, P_{F^C})$.

Lemma A1 *(Femenia, Marí, Oviedo and Neme 2008) Given* $M = (D, E, \mathbf{P})$ and $F' \subseteq F$, let μ and μ' be the stable matchings for M and $M_{F'}$ respectively. Then $\#\mu' \leq \#\mu \leq \#\mu' + \#(F \setminus F')$.

The institution's responsive preference

Given a matching market M_U and a quota $q \leq \min\{n, m\}$, we denote a $S(M_U^q)$ the set of all $q-$stable matchings. We will assume that the institution has an individual preference $\succ_D$ over the set $D \cup \varphi$ and an individual preference $\succ_E$ over the set $E \cup \varphi$ and its preferences over matchings are directly connected with its preferences over workers. An institution's preference is called responsive to its individual preferences if, for any matching that differs in only one worker, the institution prefers the matching that has the most preferable worker according to the individual preferences.

In order to formalize the institution's responsive preference, we introduce the notations that follow.

For every matching μ consider $B_\mu = \{(d,e) \in D \times E : \mu(d) = e\}$.

For every $f \in D \cup E$: $\mu^{(d,e)}(f) = \begin{cases} \varphi & if \quad f \notin \{d,e\} \\ d & if \quad f = e \end{cases}$

Notice that $\mu^{(d,e)} = \mu^{\varnothing}_{(d,e)}$.

Definition A2 A preference relation R_U is a **responsive extension** of preferences $\succ_D$ and $\succ_E$ over $D \cup \{\varphi\}$ and $E \cup \{\varphi\}$ respectively, such that it satisfies the following conditions:

i) $\mu^{(d,e)} P_U \mu^{\varphi}$ if and only if $d \succ_D \varphi$ and $e \succ_E \varphi$.

ii) $\mu P_U \mu^{\varphi}$ if and only if $\mu^{(d,e)} P_U \mu^{\varphi}$ for every $(d,e) \in B_\mu$.

iii) $\mu^{(d,e)} P_U \mu^{(d,e')}$ if and only if $e \succ_E e'$.

iv) $\mu^{(d,e)} P_U \mu^{(d',e)}$ if and only if $d \succ_D d'$.

v) For every $\mu, \mu' \in M$ such that $\#\mu = \#\mu'$ and $B_\mu = B_{\mu'} \setminus \{(d',e')\} \cup \{(d,e)\}$: $\mu P_U \mu'$ if and only if $\mu^{(d,e)} P_U \mu^{(d',e')}$.

vi) For every $\mu, \mu' \in M$ such that $B_{\mu'} \subset B_\mu$ and $\mu P_U \mu^{\varphi}$, then $\mu P_U \mu'$.

vii) For every $\mu, \mu' \in M$ such that $\mu(E) = \mu'(E)$ and $\mu(D) = \mu'(D)$, then $\mu \, I_U \mu'$.

We consider a preference R_U to be responsive if there are two individual preferences $\succ_D$ and $\succ_E$ over $D \cup \varphi$ and $E \cup \varphi$ respectively, such that R_U is a responsive extension.

Remark A3 *Given two preferences* $\succ_D$ *and* $\succ_E$ *over* $D \cup \varphi$ *and* $E \cup \varphi$ *respectively, we can construct a responsive preference relation* R_U *over the set of all matchings* M *; moreover, this extension is not unique.*

The sets $T_q(M)$ and $T_{<q}(M)$

Now we will consider the model M_U^q, where R_U is a responsive preference. Without loss of generality and in order to avoid the addition of notational complexity to the model M_U^q, we assume that all the agents of sets D and E are acceptable for the institution, i.e. for every $d \in D$ and $e \in E$, we have that $d \succ_D \varphi$ and $e \succ_E \varphi$.

For every $t \in \mathrm{N}$, we can define the following subset $F^t \subseteq F$ such that $\#F^t = t$, and for every $f \in F^t$ and $f^{'} \notin F^t$ we have that $f \succ_F f^{'}$. Note that $F^1 \subseteq F^2 \subseteq ... \subseteq F^l = F$, where $\#F = l$.

Given sets $\mathbf{d} = \{1, 2, ..., \#D\}$ and $\mathbf{e} = \{1, 2, ..., \#E\}$, for every $(t_1, t_2) \in \mathbf{d} \times \mathbf{e}$, we denote $M^{(t^*, s^*)}$, the restriction of M to D^{t^*} and E^{s^*}, i.e., $M^{(t^*, s^*)} = \left(D^{t^*}, E^{s^*}, \mathbf{P}\right)$.

Given $(t^*, s^*) \in \mathbf{d} \times \mathbf{e}$, *q*, and the following sets of matchings:

$$T_q(M^{(t^*,s^*)}) = \begin{cases} S(M^{(t^*,s^*)}) & if \quad \#\mu = q \ \text{ for every } \ \mu \in S(M^{(t^*,s^*)}) \\ \varphi & \text{otherwise} \end{cases}$$

and $T_q(M) = \{\mu : \exists \quad (t^*, s^*) \text{ such that } \mu \in T_q(M^{(t^*,s^*)})\}$.

Proposition A4 *Given* $M_U = (M, R_U)$, $(t^*, s^*) \in \mathbf{d} \times \mathbf{e}$, *there exists* $K \subseteq \mathbf{d} \times \mathbf{e}$*, such that*

$$T_q(M) = \dot{\bigcup_{(t^*,s^*) \in K}} T_q(M^{(t^*,s^*)})$$

Given $(t_1, t_2) \in \mathbf{d} \times \mathbf{e}$, *q*, and the following sets of stable matchings:

$$T_{<q}(M^{(t^*,s^*)}) = \left\{ \mu \in S(M^{(t^*,s^*)}) : \#\mu < q, \text{either } \varphi P_e d \quad \text{or} \quad \varphi P_d e \right. \\ \left. \text{for every } d \in D \setminus \mu(E^{s^*}) \quad \text{and} \quad e \in E \setminus \mu(D^{t^*}) \right\}$$

And $T_{<q}(M) = \{\mu : \exists \quad (t^*, s^*) \text{ such that } \mu \in T_{<q}(M^{(t^*,s^*)})\}$.

Proposition A5 Given $M_U = (M, R_U)$, $(t^*, s^*) \in \mathbf{d} \times \mathbf{e}$, there exists $\hat{K} \subseteq \mathbf{d} \times \mathbf{e}$, such that

$$T_{<q}(M) = \dot{\bigcup_{(t^*,s^*) \in \hat{K}}} T_{<q}(M^{(t^*,s^*)}).$$

Remark A6 *The sets* K *and* $\hat{K}$ *on the previous propositions are given by:*

$K = \{(t^*, s^*) \in N : \forall \ (t_1^*, s_1^*) \neq (t_2^*, s_2^*) \quad t_1^* \leq t_2^*, s_1^* \leq s_2^*$

such that $T_q(M^{(t_1^*, s_1^*)}) \cap T_q(M^{(t_2^*, s_2^*)}) = \varphi\}$ and

$\hat{K} = \{(t^*, s^*) \in N : \forall \ (t_1^*, s_1^*) \neq (t_2^*, s_2^*) \quad t_1^* \leq t_2^*, s_1^* \leq s_2^*$

such that $T_{<q}(M^{(t_1^*, s_1^*)}) \cap T_{<q}(M^{(t_2^*, s_2^*)}) = \varphi\}$

Now, we are going to present the following results which state that the set of *q*-stable matching are non-empty.

Teorem A1 If $M_U^q, = (M, R_U, q)$ is a matching market, then $S(M_U^q) \neq \varphi$.

The following theorem is a complete characterization of the *q*-stable sets $S(M_U^q)$.

Teorem A2 If $M_U^q, = (M, R_U, q)$ is a matching market, then $S(M_U^q) = T_q(M) \cup T_{<q}(M)$.